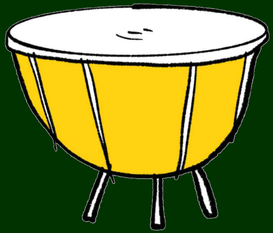


Geometry emerging from spectra

Walter van Suijlekom

Radboud Universiteit

Lorentz in October 1910

H.A. Lorentz by Jan Veth

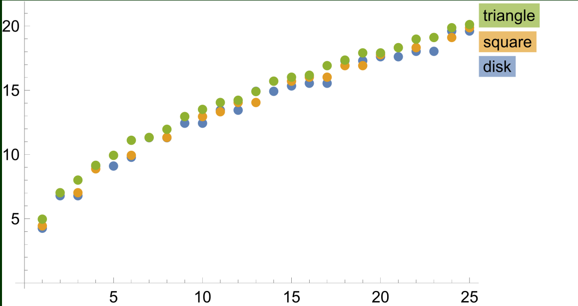
Origins of spectral geometry:

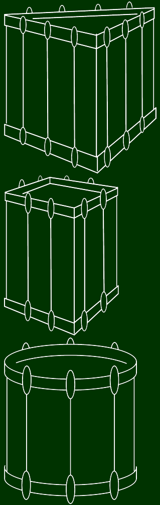
the high overtones behave inversely proportional to the volume.

Weyl in February 1911

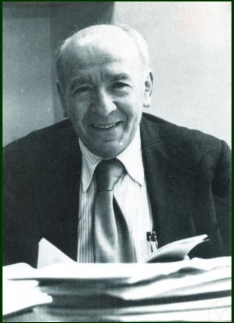
$$N(\Lambda) = \#\text{wave numbers } \leq \Lambda$$
$$\sim \frac{\Omega_d \mathrm{Vol}(M)}{d(2\pi)^d}\Lambda^d$$

Evidence by the parabolic shapes ($\sqrt{\Lambda}$):

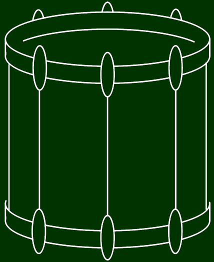




Mark Kac in 1966



"Can one hear the shape of a drum?"

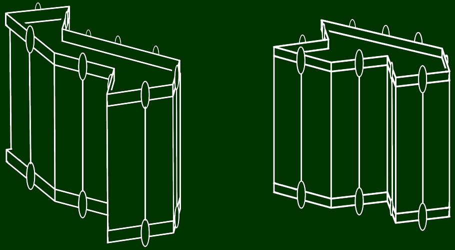


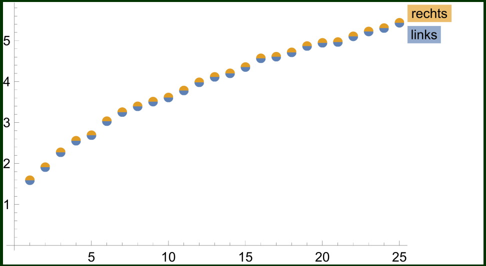
Or, more precisely, given a Riemannian manifold M, does the spectrum of wave numbers k in the Helmholtz equation

$$\boxed{\Delta_M u = k^2 u}$$

determine the geometry of M?

Isospectral drums!





... so the answer to Kac's question is **no** and more information is needed...

Spectral description of geometry: distance

Noncommutative geometry (Alain Connes)

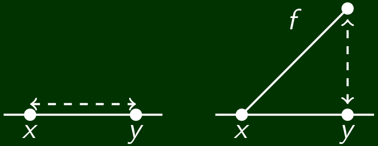
- Distance $d(x,y)$ between two points is usually defined as

  the **smallest** *of the arclengths (computed using the metric) of curves connecting x and y.*

- But it can also be defined as

  the **largest** *of differences $|f(x) - f(y)|$ for functions f with gradient $|\nabla f| \leq 1$.*

$$d(x,y) = \sup_{\|[D_M, f]\| \leq 1} |f(x) - f(y)|$$

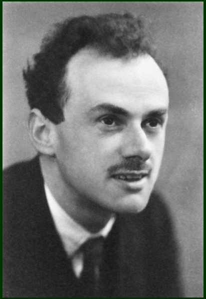


Combination $(C^\infty(M), L^2(S_M), D_M)$
allows for reconstruction of geometry

Analysis: Dirac operator

Recall that k^2 is an eigenvalue of the Laplacian in the Helmholtz equation.

- The Dirac operator is a 'square-root' of the Laplacian, so that its spectrum give the wave numbers k.
- First found by Paul Dirac in flat space, but exists on any Riemannian spin manifold M.



The circle

- The Laplacian on the circle $\mathbb{S}^1$ is given by

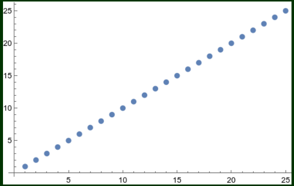
$$\Delta_{\mathbb{S}^1} = -\frac{d^2}{dt^2}; \qquad (t \in [0, 2\pi))$$

- The Dirac operator on the circle is

$$D_{\mathbb{S}^1} = -i\frac{d}{dt}$$

  with square $\Delta_{\mathbb{S}^1}$.
- The eigenfunctions of $D_{\mathbb{S}^1}$ in $L^2(S^1)$ are the complex exponential functions

$$e^{int} = \cos nt + i \sin nt; \qquad (n \in \mathbb{Z})$$



and $[D_{S^1}, f] = \frac{df}{dt}$, a bounded operator on $L^2(S^1)$ for smooth f.

The 2-dimensional torus

- Consider the two-dimensional torus $\mathbb{T}^2$ parametrized by two angles $t_1, t_2 \in [0, 2\pi)$.
- The Laplacian reads

$$\Delta_{\mathbb{T}^2} = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}.$$

- It seems difficult to construct a differential operator that squares to $\Delta_{\mathbb{T}^2}$:

$$\left(a\frac{\partial}{\partial t_1} + b\frac{\partial}{\partial t_2}\right)^2 = a^2\frac{\partial^2}{\partial t_1^2} + 2ab\frac{\partial^2}{\partial t_1 \partial t_2} + b^2\frac{\partial^2}{\partial t_2^2}$$

- This puzzle was solved by Dirac who considered complex *matrices*:

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

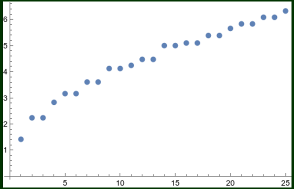
then $a^2 = b^2 = -1$ and $ab + ba = 0$

- The Dirac operator on the torus is

$$D_{\mathbb{T}^2} = \begin{pmatrix} 0 & \frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} \\ -\frac{\partial}{\partial t_1} + i\frac{\partial}{\partial t_2} & 0 \end{pmatrix},$$

  which satisfies $(D_{\mathbb{T}^2})^2 = -\frac{\partial^2}{\partial t_1^2} - \frac{\partial^2}{\partial t_2^2}$.
- The spectrum of the Dirac operator $D_{\mathbb{T}^2}$ is

$$\left\{ \pm\sqrt{n_1^2 + n_2^2} : n_1, n_2 \in \mathbb{Z} \right\};$$



  and $\|[D_{\mathbb{T}^2}, f]\| = \|f\|_{\text{Lip}}$.

More generally, a Dirac operator exists on spin manifolds as a differential operator acting in $L^2(S_M)$ and square $D_M^2 = \Delta_M + \frac{1}{4}\kappa$.

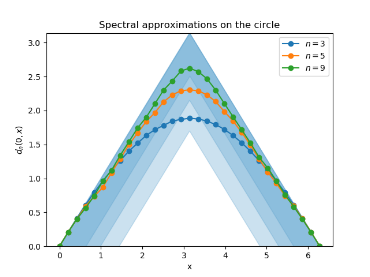
Example: spectral truncation of the circle

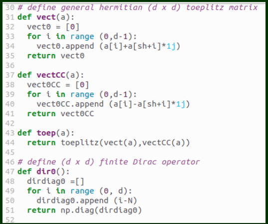
- Eigenvectors of D_{S^1} are **Fourier modes** $e_k(t) = e^{ikt}$ for $k \in \mathbb{Z}$
- **Orthogonal projection** P_n onto $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, \ldots, e_n\}$
- The space $P_n C(S^1) P_n$ is an **operator system**
- Any such $T = P_n f P_n$ can be written as a **Toeplitz matrix**

$$P_n f P_n \sim \left(t_{k-l}\right)_{kl} = \begin{pmatrix} t_0 & t_{-1} & \cdots & t_{-n+2} & t_{-n+1} \\ t_1 & t_0 & t_{-1} & & t_{-n+2} \\ \vdots & t_1 & t_0 & \ddots & \vdots \\ t_{n-2} & & \ddots & \ddots & t_{-1} \\ t_{n-1} & t_{n-2} & \cdots & t_1 & t_0 \end{pmatrix}$$

- Evaluation at points of S^1 are replaced by certain positive linear functionals ϕ_x (states) from $P_n C(S^1) P_n$ to $\mathbb{C}$.

Distance function for spectral truncations of the circle





```
# define general hermitian (d x d) toeplitz matrix
def vect(a):
  vect0 = [0]
  for i in range (0,d-1):
    vect0.append (a[i]+a[sh+i]*1j)
  return vect0

def vectCC(a):
  vect0CC = [0]
  for i in range (0,d-1):
    vect0CC.append (a[i]-a[sh+i]*1j)
  return vect0CC

def toep(a):
  return toeplitz(vect(a),vectCC(a))

# define (d x d) finite Dirac operator
def dir0():
  dirdiag0 =[]
  for i in range (0, d):
    dirdiag0.append (i-N)
  return np.diag(dirdiag0)
```

$$d(\phi_x,\phi_y) \leq d_{S^1}(x,y) \leq d(\phi_x,\phi_y) + 2\gamma_n$$

And more examples include (quantum) fuzzy spheres, Fourier truncations, truncations of tori (Leimbach–vS23, RU) ...

Courses at Radboud Uni

Currently teaching:

- Inleiding Wiskunde
- Lie Groups (MM, with Erik Koelink)
- Noncommutative Geometry (MM)

and other relevant courses:

- (Intro to) Functional Analysis
- Operator Algebras
- Differential/Riemannian Geometry